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Comparison of Numerical Solutions of Time-Fractional Reaction-Diffusion Equations

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ABSTRACT

Nonlinear phenomena play a crucial role in applied mathematics and physics. Although it is very easy for us now to find the solutions of nonlinear problems by means of computers, it is still rather difficult to solve nonlinear problems either numerically or theoretically. One of the most famous of the nonlinear fractional partial differential equations which called the time-fractional reaction-diffusion equation in this paper, we compare numerical solutions for time-fractional reaction-diffusion equation using variation iteration, homotopy perturbation, adomian decomposition and differential transform methods. The fractional derivatives are described in the Caputo sense. The methods in applied mathematics can be used as alternative methods for obtaining analytic and approximate solutions for different types of fractional partial differential equations. The approach rest mainly on two-dimensional differential transform method which is one of the most efficient from approximate methods. The method can easily be applied to many linear and nonlinear problems and is capable of reducing the size of computational work. An example is given to demonstrate the effectiveness of the present method.

Keywords: Fractional differential equation, differential transform method, time-fractional reaction-diffusion equations.

1. INTRODUCTION

There is a long-standing interest in extending the classical calculus to non-integer orders (Oldham and Spanier (1974); Podlubny (1999); Caputo (1967)) because fractional differential equations are suitable models for many physical problems. The development of this generalized calculus, however, has been consistently hampered because many fractional differential equations are nonlinear and have no exact analytical solutions that can be likened to the classical case. Many numerical schemes have been proposed over the years to approximate the solutions of fractional equations, for example, the Adomian Decomposition Method (ADM) (Yu *et al.* (2008); Jafari *et al.* (2007)), the Variation Iteration Method (VIM) (Momani (2007); Odibat and Momani (2006); Yulita *et al.* (2009)), the Homotopy Perturbation Method (HPM) (Momani and Odibat (2007), the Differential Transform Method (DTM) (Momani *et al.* (2007); Oturanc *et al.* (2008).

Consider the nonlinear initial-boundary value time-fractional reaction-diffusion parabolic problems

$$D_t^{\alpha} = Du_{xx} + mu\left(1-u\right) \tag{1}$$

subject to initial conditions

$$u\left(x,0\right) = f\left(x\right) \tag{2}$$

where u is a function of x and t, $f\left(x\right)$ is a known analytic function. These equations were first introduced by Fisher as a model for the propagation of a mutant gene. It has wide application in the fields of logistic population growth, flame propagation, europhysiology, autocatalytic chemical reactions, branching Brownian motion processes, and nuclear reactor theory (Fisher (1937); Murray (1977); Britton (1986); Frank (1955)).

In this paper, we compare numerical solutions of time-fractional reaction-diffusion equation using homo-topy perturbation (Momani and Odibat (2007a)), variational iteration (Odibat and Momani (2008b)), adomian decomposition (Odibat and Momani (2008b)) and differential transform methods. We solve time-fractional reaction-diffusion equation by the differential transform method. The main advantage of the method is the fact that it provides its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed terms.

2. BASIC DEFINITIONS

In this section, we present some basic definitions and properties of the fractional calculus (Podlubny (1999); Caputo (1967)).

Definition 1 *A real function* $f(x), x>0$ *is said to be in the space* C_{μ}, $\mu \in R$ *if there exists a real number* $(p>\mu)$, *such that* $f(x)=x^{p} f_{1}(x)$, *where* $f_{1}(x) \in C[0,\infty)$ *and it said to be in the space*

$$C_{\mu}^{m} \text{ iff } f^{m} \in C_{\mu},\ m \in N.$$

Definition 2 *The Riemann-Liouville fractional integral operator of order* $\alpha \geq 0$ *of a function* $f \in C_{\mu}, \mu \geq -1$ *is defined as*

$$J_{0}^{\upsilon} f(x)=\frac{1}{\Gamma(\upsilon)} \int_{0}^{x}(x-t)^{\upsilon-1} f(t)\, dt, \quad \upsilon>0,$$
$$J^{0} f(x)=f(x).$$

It has the following properties:

For $f \in C_{\mu}, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma>1$:

1. $J^{\alpha} J^{\beta} f(x)=J^{\alpha+\beta} f(x)$,
2. $J^{\alpha} J^{\beta} f(x)=J^{\beta} J^{\alpha} f(x)$,
3. $J^{\alpha} x^{\gamma}=\dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann-Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for the physical problems of the real world since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations.

Definition 3 *The fractional derivative of* $f(x)$ *in the Caputo sense is defined as*

$$D_{*}^{\upsilon} f(x)=J_{a}^{m-\upsilon} D^{m} f(x)=\frac{1}{\Gamma(m-\upsilon)} \int_{0}^{x}(x-t)^{m-\upsilon-1} f^{(m)}(t)\, dt,$$

for $m-1<\upsilon<m,\ m \in N,\ x>0,\ f \in C_{-1}^{m}$.

Lemma 4 *If* $m-1<\alpha<m,\ m\in N$ and $f\in C_\mu^m, \mu\geq -1,$ *then*

$$D_*^\alpha J^\alpha f(x)=f(x),$$

$$J^\alpha D_*^\upsilon f(x)=f(x)-\sum_{k=0}^{m-1} f^k\left(0^+\right)\frac{x^k}{k!}, x=0.$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem.

Definition 5 *For* m *to be the smallest integer that exceeds* α, *the Caputo time-fractional derivative operator of order* $\alpha>0$ *is defined as*

$$D_{*x}^\alpha u(x,t)=\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-\xi)^{m-\alpha-1}\frac{\partial^m u(x,\xi)}{\partial \xi^m}d\xi, & for\ m-1<\alpha<m,\\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & for\ \alpha=m\in N\end{cases}$$

and the space-fractional derivative operator of order $\beta>0$ *is defined as*

$$D_{*x}^\alpha u(x,t)=\frac{\partial^\beta u(x,t)}{\partial x^\beta}=\begin{cases}\dfrac{1}{\Gamma(m-\beta)}\displaystyle\int_0^x (x-\theta)^{m-\beta-1}\frac{\partial^m u(\theta,t)}{\partial \theta^m}d\theta, & for\ m-1<\beta<m,\\ \dfrac{\partial^m u(x,t)}{\partial x^m}, & for\ \beta=m\in N.\end{cases}$$

3. DIFFERENTIAL TARNSFORM METHOD

The DTM is applied to the solution of electric circuit problems. The DTM is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the DTM obtains a polynomial series solution by means of an iterative procedure. The method is well addressed in Momani *et al.* (2007).

Consider a function of two variables $u(x,y)$ and suppose that it can be represented as a product of two single-variable functions, i.e., $u(x,y)=f(x)g(y)$. Based on the properties of generalized two-dimensional differential transform (Bildik et al. (2006); Abdel-Halim (2008)), the function $u(x,y)$ can be represented as

$$\begin{aligned} u(x,y) &= \sum_{k=0}^{\infty} F_{\alpha}(k)(x-x_0)^{k\alpha} \sum_{h=0}^{\infty} G_{\beta}(h)(y-y_0)^{h\beta} \\ &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha\beta}(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta} \end{aligned} \quad (3)$$

where $0<\alpha,\ \beta\le 1,\ \ U_{\alpha\beta}(k,h)=F_{\alpha}(k)G_{\beta}(h)$ is called the spectrum of $u(x,y)$. The generalized two-dimensional differential transform of the function u(x, y) is given by

$$U_{\alpha,\beta}(k,h)=\frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D_{*x_0}^{\alpha}\right)^{k}\left(D_{*y_0}^{\beta}\right)^{h}u(x,y)\right]_{(x_0,y_0)}, \quad (4)$$

where $\left(D_{x_0}^{\alpha}\right)^{k}=D_{x_0}^{\alpha}D_{x_0}^{\alpha}\cdots D_{x_0}^{\alpha},k$ - times. In case of $\alpha=1$ and $\beta=1$ the generalized two-dimensional differential transform (3) reduces to the classical two-dimensional differential transform (Momani and Odibat (2008)).

The operators in two-dimensional differential transformation method (Momani and Odibat (2008)):

Let $U_{\alpha,\beta}(k,h),V_{\alpha,\beta}(k,h)$ and $W_{\alpha,\beta}(k,h)$ be the differential transformations of the functions $u(x,y),v(x,y)$ and $w(x,y)$:

If $u(x,y)=u(x,y)\pm w(x,y)$, then $U_{\alpha,\beta}(k,h)=V_{\alpha,\beta}(k,h)\pm W_{\alpha,\beta}(k,h)$,

If $u(x,y)=av(x,y),a\in R$, then $U_{\alpha,\beta}(k,h)=aV_{\alpha,\beta}(k,h)$,

If $u(x,y)=v(x,y)w(x,y)$, then $U_{\alpha,\beta}(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}V_{\alpha,\beta}(r,h-s),W_{\alpha,\beta}(k-r,s)$, If $u(x,y)=(x-x_0)^{n\alpha}(y-y_0)^{m\beta}$, then $U_{\alpha,\beta}(k,h)=\delta(k-n)\delta(h-m)$,

If $u(x,y)=v(x,y)w(x,y)q(x,y)$, then

$$U_{\alpha,\beta}(k,h)=\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{t=0}^{h}V_{\alpha,\beta}(r,h-s-p)W_{\alpha,\beta}(t,s)Q_{\alpha,\beta}(k-r-t,p),$$

If $u(x,y)=D_{x_0}^{\alpha}v(x,y),\ 0<\alpha\leq 1,$ then $U_{\alpha,\beta}(k,h)=\dfrac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}V_{\alpha,\beta}(k+1,h),$

If $u(x,y)=f(x)g(y)$ and the function $f(x)=x^{\lambda}h(x)$, where $\lambda>-1, h(x),$ has the generalized Taylor series expansion $h(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^{\alpha k}$, and (Momani and Odibat (2008)), $\beta<\lambda+1$ and α arbitrary or $\beta\geq\lambda+1$, α arbitrary and $a_n=0$ for $n=0,1,...,m-1$, where $m-1<\beta\leq m.$

Then the generalized differential transform (4) becomes

$$U_{\alpha,\beta}(k,h)=\frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[D_{*x_0}^{\alpha k}\left(D_{*y_0}^{\beta}\right)^{h}u(x,y)\right]_{(x_0,y_0)},$$

If $u(x,y)=D_{x_0}^{\gamma}v(x,y),\ m-1<\gamma\leq m$ and $v(x,y)=f(x)g(y)$, then

$$U_{\alpha,\beta}(k,h)=\frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)}V_{\alpha,\beta}(k+\gamma/\alpha,h).$$

If $u(x,y,t)=D_{*x_0}^{\alpha}v(x,y,t),\ 0<\alpha\leq 1$ then

$$U_{\alpha,\beta,\gamma}(k,h,m)=\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}V_{\alpha,\beta,m}(k+1,h,m).$$

If $u(x\cdot y)=a(x,y)\dfrac{\partial v(x,y)}{\partial x}$ then

$$U(k,h)=\sum_{i=0}^{k}\sum_{j=0}^{h}(k-i+1)A(i,j)U(k-i+1,h-j).$$

The proofs of t he some properties can be found in Momani and Odibat (2008).

4. APPLICATION

Let us consider following time fractional reaction-diffusion equation is called as Fisher equation

$$D_{*t}^{\alpha}u(x,t)=u_{xx}(x,t)+6u(x,t)(1-u(x,y)),\ t>0,\ x\in R,\ 0<\alpha\leq 1, \tag{5}$$

with initial condition

$$u(x,0)=\frac{1}{\left(1+e^{x}\right)^{2}} \tag{6}$$

where $u=u(x,t)$ is a function of the variables x and t. Then, by using the basic properties of the differential transformation, we can find the transformed form of equation (5) as

$$\begin{aligned}\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}U_{\alpha,1}(k,h+1)&=(k+1)(k+2)U_{\alpha,1}(k+2,h)+6U_{\alpha,1}(k,h)\\&-6\sum_{r=0}^{k}\sum_{s=0}^{h}U_{\alpha,1}(r,h-s)U_{\alpha,1}(k-r,s).\end{aligned} \tag{7}$$

Using the initial condition (6), we have

$$\mathrm{U}(0,0)=\frac{1}{4},\ \mathrm{U}(1,0)=-\frac{1}{4},\mathrm{U}(2,0)=\frac{1}{4},\mathrm{U}(3,0)=\frac{1}{48},\mathrm{U}(4,0)=\frac{1}{96},\ldots \tag{8}$$

Now, substituting (8) into (7), we obtain the following $U(x,t)$ values successively

$$u(x,t)=\frac{1}{4}-\frac{1}{4}x+\frac{1}{16}x^2+\frac{1}{48}x^3+\left(\frac{5}{4}-\frac{5}{8}x-\frac{5}{16}x^2\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$
$$+\left(\frac{25}{16}+\frac{25}{16}x\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{125}{8}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots \tag{9}$$

which is the solution of (5) in series form.

Finally the differential inverse transform of $U(x,t)$ gives

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{\alpha\beta}(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta}=\sum_{k=0}^{\infty}\frac{\partial k}{\partial t^k}\left(\frac{1}{\left(1+e^{x-5t}\right)^2}\right)_{t=0}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}.$$

We, therefore, obtain

$$u(x,t)=\left(\frac{1}{\left(1+e^{x-5t}\right)^2}\right)$$

which is the exact solution.

Remark 6 $\alpha=1$ *is the only value of* α *for which the exact solution* $u(x,t)=\left(\frac{1}{\left(1+e^{x-5t}\right)^2}\right)$ *is known and the above approximate solutions are in good agreement with these exact values.*

TABLE 1: Numerical values when $\alpha = 1$ for Equation (5)

t	x	U_{VIM}	u_{HPM}	u_{ADM}	u_{DTM}	$u_{Exact\ Solution}$
0.1	0.25	0.315940	0.315940	0.317948	0.316080	0.316042
0.1	0.50	0.249926	0.249926	0.200500	0.250000	0.250000
0.1	0.75	0.191606	0.191606	0.190964	0.191731	0.191689
0.1	1.0	0.142411	0.142411	0.140979	0.143229	0.142537
0.2	0.25	0.459320	0.459320	0.481199	0.463867	0.461284
0.2	0.50	0.368420	0.368420	0.396941	0.388020	0.387456
0.2	0.75	0.315478	0.315478	0.315266	0.316080	0.316042
0.2	1.0	0.249092	0.249092	0.241175	0.250000	0.250000
0.3	0.25	0.591179	0.591179	0.681440	0.619466	0.604195
0.3	0.50	0.527635	0.527635	0.581861	0.541666	0.534447
0.3	0.75	0.459719	0.459719	0.475833	0.463867	0.461284
0.3	1.0	0.387025	0.387025	0.372917	0.388020	0.387456

It is to be noted that only the fourth-order term of the variational iteration solution (Odibat and Momani (2008b)) and homotopy perturbation solution (Momani and Odibat (2007a)) only three terms of the decomposition series (Odibat and Momani (2008b)) and transform solution were used in evaluating the approximate solutions for Table 1.

5. CONCLUSION

In this paper, we compare numerical solutions for time-fractional reaction-diffusion equation using homotopy perturbation (Odibat and Momani (2008b)), variational iteration (Momani and Odibat (2007a)), adomian decomposition (Momani and Odibat (2007a)) and differential transform methods. We solve time-fractional reaction-diffusion equation by the differential transform method. The main advantage of the method is the fact that it provides its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed terms.

One example shows that the differential transform method is a powerful mathematical tool to solving time fractional reaction-diffusion equation. It is also a promising method to solve other nonlinear equations. This method solves the problem without any need to discretization of the variables, therefore, it is not affected by computation round off errors and one does not face the need of large computer memory and time. In our work, we made use of the Maple Package to calculate the series obtained from the differential transform method.

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